INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 8, 759–768 (1988)

ON THE SELF-INDUCED MOTION OF VORTEX SHEETS

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SUMMARY

An accurate numerical scheme has been devised to study the self-induced motion of an infinitely thin, free vortex sheet of finite span in an unbounded, inviscid, incompressible fluid. The new numerical scheme has been tested against two vortex sheet problems for which exact solutions have also been obtained. The agreement between the numerical and exact solutions is excellent. The scheme has been further tested against two more examples for which analytical solutions for small times were available. Here too the agreement is excellent.

KEY WORDS Vortex sheets Inviscid flow Incompressible flow

1. INTRODUCTION

The calculation of the self-induced motion of a free, infinitely thin, continuous vortex sheet of finite span in an unbounded, inviscid, incompressible fluid at rest at infinity is a fundamental problem in fluid dynamics. The simplicity of its mathematical statement is in sharp contrast to its tenacity to resist an analytical solution. Several basic mathematical questions regarding the problem have yet to be satisfactorily answered. For example, do such vortex sheets roll up? If so, under what circumstances? Or, do such vortex sheets have smooth motion for all times or do the sheets develop kinks after a finite time interval? Or, what are the permissible vorticity distributions on the sheet for the problem to be meaningful? And so on.

The absence of clear-cut answers to these questions, and the lack of a general analytical method, has prompted various authors at different times to find a numerical solution by boldly disregarding the finer mathematical aspects of the problem. Thus we find the very simple point vortex method of Westwater[1] which in its time was considered a major success in aerodynamics. Only later did Takami[2] and Moore[3] find that the method had serious problems and was basically unreliable. However, the inherent appeal of the method due to its simplicity was so strong that several *ad hoc* schemes[3–5] were devised to 'improve' the method. None of these 'improvements' have any real theoretical basis and they only serve to distort the original statement of the problem. A detailed criticism of the method has recently been provided by Bera.[6] There it is shown that the source of difficulties with the point vortex method is that it introduces new singularities into the problem and removes the existing ones. Thus, in essence, the point vortex method solves, not the original problem, but an entirely different one.

A more sophisticated method was proposed by Fink and Soh[7] with apparently better theoretical justification. But here too it can be shown that the method fails to correctly account for the singularities of the original problem, and therefore their results are, once again, in considerable error.

0271–2091/88/070759–10$05.00

Received 24 April 1987
Revised 9 September 1987

There is a strongly held 'belief' among fluid dynamicists that vortex sheets naturally roll up into spiral vortices. The 'success' of Westwater was that his crude calculations showed his initially plane vortex sheet to roll up subsequently in the anticipated manner, thus providing the strong incentive to others to devise 'improved' *ad hoc* schemes of the point vortex method even when the reliability of the method was put in doubt. The Fink and Soh method also showed the Westwater vortex sheet to roll up. In retrospect this is indeed interesting, since, apparently unknown to all these authors, an exact solution to the Westwater problem exists and it does not show any roll up! Instead, the vortex sheet moves perpendicular to itself at constant speed!

It is therefore clear that a sound numerical scheme for the problem should, in principle, properly account for the singularities contained in the problem. We shall look at these singularities in Section 2. In Section 3 we provide the analytical test cases against which the numerical scheme devised in this paper will be tested. In Section 4 we derive the numerical scheme and in Section 5 test it against the analytical test cases. Section 6 provides the conclusions.

However, before we proceed to the next section, a few remarks in a wider context are in order. It is generally difficult to imagine any interesting fluid flow without vorticity. The two-dimensional problem studied in this paper is one where the vorticity distribution is on an infinitesimally thin sheet of finite span placed in an otherwise unbounded uniform flow. For a class of vorticity distributions on this sheet an exact numerical method of calculating its dynamics has been obtained. In general, for other spatial vorticity distributions (e.g. infinitesimally thin vortex sheets of infinite span, vortex blobs of finite size, etc. in two dimensions, or distributions in three dimensions) considerable work remains to be done in devising accurate numerical methods for describing their temporal behaviour. In this regard the interested reader may wish to consult the recent important contributions of Pullin and Grimshaw,[8] Anderson,[9] Krasny[10,11] and DiPerna and Majda[12] and the references cited therein. They provide an insight into the very difficult nature of the task at hand.

2. THE MATHEMATICAL SINGULARITIES

Consider a two-dimensional free vortex sheet whose vorticity vector is perpendicular to the x–y plane and whose Trefftz plane cross-section at some time t lies on a smooth curve $C(t)$ defined by $x=X(s,t)$, $y=Y(s,t)$, where $X(s,t)$, $Y(s,t)$ are the Cartesian co-ordinates of a point on $C(t)$ located at the arc length s (see Figure 1). Let $G(s,t)$ be the circulation per unit length on $C(t)$. The self-induced velocity with which the vortex sheet convects itself is then given by

$$u(s,t) = -\frac{\partial}{\partial y}\int_{C(t)} \frac{1}{2\pi}\ln|\mathbf{r}(s,t)-\mathbf{r}(\sigma,t)|G(\sigma,t)\,d\sigma, \tag{1a}$$

$$v(s,t) = \frac{\partial}{\partial x}\int_{C(t)} \frac{1}{2\pi}\ln|\mathbf{r}(s,t)-\mathbf{r}(\sigma,t)|G(\sigma,t)\,d\sigma, \tag{1b}$$

where (x, y) are the Cartesian co-ordinates in the plane of the flow, $u(s,t)$, $v(s,t)$ are the respective x-, y-velocity components on $C(t)$, σ is the dummy integration variable corresponding to s and

$$\mathbf{r} = [X(s,t),\ Y(s,t)], \qquad |\mathbf{r}| = [X^2(s,t)+Y^2(s,t)]^{1/2}. \tag{2}$$

The problem now is to determine the motion (u, v) and shape $C(t)$ of the vortex sheet, given its location, shape and vorticity distribution at some initial instant $t=t_0$.

On the vortex sheet the integrals in equations (1) are singular at $s=\sigma$ and therefore they must be interpreted in their Cauchy principal value sense. Also, depending upon $G(s,t)$, $[u(s,t), v(s,t)]$ may, in turn, contain other singularities. However, according to a lemma due to Lighthill,[13] if

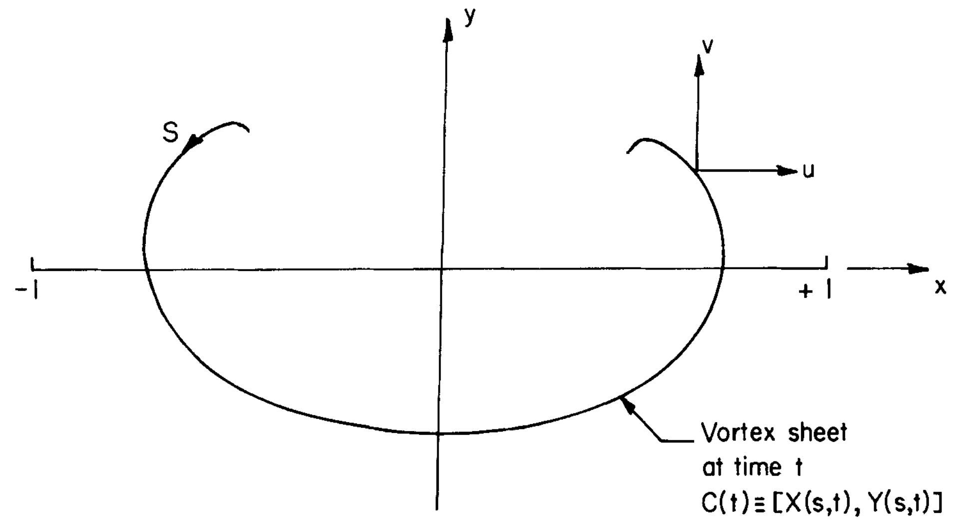

Figure 1. Nomenclature for vortex sheet in the Trefftz plane

$(1-\xi^2)^{1/2}h(\xi)$ is regular in $-1 \leqslant \xi \leqslant 1$, then so is

$$\int_{-1}^{1} \frac{h(\xi)}{x-\xi} \mathrm{d}\xi \quad \text{in} \quad -1 \leqslant x \leqslant 1.$$

Now a vortex sheet is composed of vortex filaments placed side by side in continuous succession and these filaments retain their independent material identity at all times. Since x and y are related on $C(t)$ and the sheet cannot cut through itself, it follows that by suitably orienting a Cartesian co-ordinate system at some instant $t=t_0$ we may assign to each vortex filament a unique and single value of the co-ordinate x. Thus x can be treated as a material or Lagrangian co-ordinate and $s(t)$ will monotonically increase (or decrease) with x on $C(t)$. Also, since the strength of a vortex filament remains invariant with time, we can replace integrals in $G(s, t)\,\mathrm{d}s$ by integrals of $G(x)\,\mathrm{d}x$, where $G(x)$ is not a function of time. For convenience x can be normalized so that it ranges, sheet edge to sheet edge, from $x=-1$ to $x=1$ on $C(t)$. If we now assume that $C(t)$ remains a smooth and continuous curve in the time interval of interest, then its defining functions $X(s, t)$ and $Y(s, t)$ can each be represented as polynomials of x. Consequently, by using Lighthill's lemma we can show that if $(1-x^2)^{1/2}G(x)$ is regular in $-1 \leqslant x \leqslant 1$, then $u(s, t)$ and $v(s, t)$ with which the vortex sheet convects itself are regular on $C(t)$.

By 'regular' we shall mean here that the function is continuous and all its derivatives exist in its domain of definition. On the other hand, we shall call a function 'singular' if the function or any of its derivatives become infinite in its domain of definition.

To the best of the author's knowledge, for any vorticity distribution on a smooth curve $C(t)$ for which $F=(1-x^2)^{1/2}G(x)$ is not regular in $-1 \leqslant x \leqslant 1$, (u, v) will generally show singular behaviour at the vortex sheet edges. (Two such examples are given in the Appendix.) We shall therefore restrict our attention to only those vorticity distributions for which F is regular in $-1 \leqslant x \leqslant 1$ in the sense of Lighthill's lemma.

3. ANALYTICAL TEST CASES

From Section 2 it would appear that the only vorticity distributions worth studying are those for which $(1-x^2)^{1/2}G(x)$ is a regular function, since by Lighthill's lemma they would then produce a sheet convection velocity which is regular everywhere on it. The four analytical test cases that we cite below belong to this class.

(a) $(1-x^2)^{1/2}G(x)=a_1x+a_0$ *on* $-1\leqslant x\leqslant 1$, $y=0$ *at* $t=0$

For this vorticity distribution the motion of the sheet is given by

$$X(s,t)=x, \qquad Y(s,t)=-\tfrac{1}{2}a_1t, \tag{3}$$

$-1\leqslant x\leqslant 1$, $t\geqslant 0$. That is, the vortex sheet moves with a constant linear speed $-\frac{1}{2}a_1$ in the y-direction. The Westwater example is obtained by putting $a_0=0$. The special case when $a_1=0$, $a_0\neq 0$ produces a vortex sheet which remains stationary.

(b) $(1-x^2)^{1/2}G(x)=a_2(1-2x^2)+a_0$ *on* $-1\leqslant x\leqslant 1$, $y=0$ *at* $t=0$

For this vorticity distribution the motion of the sheet is given by

$$X(s,t)=x\cos(a_2t), \qquad Y(s,t)=x\sin(a_2t), \tag{4}$$

$-1\leqslant x\leqslant 1$, $t\geqslant 0$. That is, the vortex sheet rotates with a constant angular speed a_2 about the origin $(x, y)=(0, 0)$ for all times. Once again the special case $a_2=0$, $a_0\neq 0$ produces a vortex sheet which remains stationary. Another special case when $a_2=a_0\neq 0$ is reported in Batchelor.[14]

(c) $(1-x^2)^{1/2}G(x)=3x(1-x^2)$ *on* $-1\leqslant x\leqslant 1$, $y\geqslant 0$ *at* $t=0$

For this case van Dyke has given a series solution in t for the motion of the vortex sheet (and quoted in Takami[2]) which is noted below:

$$\begin{aligned} X(s,t)&=x-\tfrac{9}{4}x(x^2-\tfrac{1}{2})t^2-\tfrac{27}{8}x(x^4-\tfrac{5}{4}x^2+\tfrac{5}{16})t^4-\ldots, \\ Y(s,t)&=\tfrac{3}{2}(x^2-\tfrac{1}{2})t-\tfrac{9}{8}(x^4-\tfrac{1}{4})t^3-\ldots. \end{aligned} \tag{5}$$

(d) $(1-x^2)^{1/2}G(x)=x$ *on* $-1\leqslant x\leqslant 1$, $y=\varepsilon(1-x^2)$ *at* $t=0$

This is an example of a vortex sheet which is curved in the x–y plane at $t=0$. For very small times the sheet geometry evolves approximately as

$$\begin{aligned} X(s,t)&=x+\tfrac{1}{2}[-2\varepsilon x+2\varepsilon^3(4x^3+x)-\varepsilon^5(32x^5+13x^3+\tfrac{9}{4}x) \\ &\quad+\varepsilon^7(128x^7+60x^5+24x^3+\tfrac{15}{6}x)-\ldots]t+\ldots, \\ Y(s,t)&=\varepsilon(1-x^2)+\tfrac{1}{2}[-1+\varepsilon^2(4x^2+\tfrac{1}{2})-\varepsilon^4(16x^4+\tfrac{11}{2}x^2+\tfrac{3}{8}) \\ &\quad+\varepsilon^6(64x^6+\tfrac{57}{2}x^4+\tfrac{33}{4}x^2+\tfrac{15}{48})-\ldots]t+\ldots. \end{aligned}$$

The exact analytical solutions of the test cases (a) and (b), in their general form, appear to be new.

4. THE NUMERICAL SCHEME

The basis of the numerical scheme we shall now devise to calculate the motion of the vortex sheet is Stark's algorithm. Stark[15] developed an elegant numerical algorithm to evaluate the Cauchy integral

$$I(x)=\int_{-1}^{1}\frac{w(\xi)f(\xi)}{\xi-x}\mathrm{d}\xi,\quad -1<x<1.$$

His quadrature rule for this can be written in the form

$$\int_{-1}^{1}\frac{w(\xi)f(\xi)}{\xi-x_j}\mathrm{d}\xi=\sum_{i=1}^{N}\frac{e_i w(\xi_i)f(\xi_i)}{\xi_i-x_j},\tag{6}$$

which is exact at the points $x=x_j$ under the following restrictions:

(a) $f(x)$ can be represented as a polynomial of degree $\leqslant 2N$.
(b) $w(x)$ is positive and integrable but not necessarily regular.
(c) The points $x=\xi_i$, $i=1, 2, \ldots, N$, are the N zeros of the polynomial $P_N(x)$ of degree N in the system $\{P_i(x)\}$ of orthogonal polynomials with $w(x)$ as the weight function and the range of integration as $-1<x<1$.
(d) The points $x=x_j$ are the zeros of the function

$$Q_N(x)=-\frac{1}{2}\int_{-1}^{1}\frac{w(\xi)P_N(\xi)}{\xi-x}\mathrm{d}\xi.$$

(e) The coefficients (the so called weights of the quadrature rule) e_i are defined by

$$e_i=-2Q_N(\xi_i)/[w(\xi_i)P'_N(\xi_i)],$$

where $P'_N(\xi_i)$ is the derivative of $P_N(x)$ at $x=\xi_i$.

We may now note that if the weight function $w(x)$ is any of

$$\begin{aligned}&\text{(a)}\ w(x)=(1-x^2)^{1/2}\\&\text{(b)}\ w(x)=(1-x^2)^{-1/2}\\&\text{(c)}\ w(x)=(1-x)^{1/2}(1+x)^{-1/2}\\&\text{(d)}\ w(x)=(1-x)^{-1/2}(1+x)^{1/2}\end{aligned}\tag{7}$$

and $f(x)$ is a regular function, then $(1-x^2)^{1/2}\,w(x)f(x)$ is regular. Further, from Lighthill's lemma the integral $I(x)$ is also regular in $-1\leqslant x\leqslant 1$.

A further point to note is that the four weight functions (a)–(d) can all be written in the form $(1-x^2)^{-1/2}\psi(x)$, where $\psi(x)$ is a regular function which can be multiplied by $f(x)$ to produce a new regular function. Thus we shall henceforth concentrate our attention only on the weight function $w(x)=(1-x^2)^{-1/2}$. For this we can show that ξ_i, x_j are the zeros of the Tchebycheff polynomials of the first and second kinds respectively. Thus

$$\xi_i=\cos[(2i-1)\pi/2N],\quad i=1, 2, \ldots, N,\tag{8}$$

$$x_j=\cos(j\pi/N),\qquad j=1, 2, \ldots, N-1\tag{9}$$

and the weights of the quadrature rule are

$$e_i=\frac{\pi}{N}(1-\xi_i^2)^{1/2}.\tag{10}$$

To adapt Stark's algorithm to solve equations (1), we make the assumption that $C(t)$ is a smooth curve in the time interval $t_0\leqslant t\leqslant t_f$, where t_f is some final time up to which we wish to calculate the motion of the vortex sheet. If this assumption is valid, then $X(s, t)$, $Y(s, t)$ will each have a polynomial representation in the co-ordinate x in the interval $-1 \leqslant x \leqslant 1$. Thus

$$\begin{aligned} X(x,t)-X(\xi,t)&=(x-\xi)G_1(x,\xi,t),\\ Y(x,t)-Y(\xi,t)&=(x-\xi)G_2(x,\xi,t) \end{aligned} \tag{11}$$

and

$$\begin{aligned} R^2&=[X(x,t)-X(\xi,t)]^2+[Y(x,t)-Y(\xi,t)]^2\\ &=(x-\xi)^2(G_1^2+G_2^2), \end{aligned} \tag{12}$$

where G_1, G_2 are regular functions of x, ξ, t. Equations (1) may now be recast in the form

$$u(x,t)=\int_{-1}^{1}\frac{G(\xi)H_1(x,\xi,t)}{\xi-x}\,d\xi, \qquad v(x,t)=\int_{-1}^{1}\frac{G(\xi)H_2(x,\xi,t)}{\xi-x}\,d\xi, \tag{13}$$

where

$$H_1=\frac{1}{2\pi}\left(\frac{G_2}{G_1^2+G_2^2}\right), \qquad H_2=-\frac{1}{2\pi}\left(\frac{G_1}{G_1^2+G_2^2}\right). \tag{14}$$

Now, since a vortex sheet cannot cut through itself at any time, R^2 cannot be zero except at $x=\xi$. This implies that

$$G_1^2+G_2^2>0 \tag{15}$$

and hence H_1 and H_2 are non-singular and well behaved functions on the vortex sheet. We may therefore extend the Stark algorithm to the above integrals and write, on the vortex sheet,

$$\begin{aligned} u(x_j,t)&=\sum_{i=1}^{N} e_i G(\xi_i)H_1(x_j,\xi_i,t)/(\xi_i-x_j),\\ v(x_j,t)&=\sum_{i=1}^{N} e_i G(\xi_i)H_2(x_j,\xi_i,t)/(\xi_i-x_j) \end{aligned} \tag{16}$$

or, alternatively,

$$\begin{aligned} u(x_j,t)&=-\frac{1}{2\pi}\sum_{i=1}^{N} e_i G(\xi_i)[Y(x_j,t)-Y(\xi_i,t)]/R_{ij}^2,\\ v(x_j,t)&=\frac{1}{2\pi}\sum_{i=1}^{N} e_i G(\xi_i)[X(x_j,t)-X(\xi_i,t)]/R_{ij}^2, \end{aligned} \tag{17}$$

where

$$R_{ij}^2=[X(x_j,t)-X(\xi_i,t)]^2+[Y(x_j,t)-Y(\xi_i,t)]^2.$$

The functions $X(x_j,t)$, $Y(x_j,t)$ are obtained from the previous time step; that is,

$$\begin{aligned} X(x_j,t)&=X(x_j,t-\Delta t)+u(x_j,t-\Delta t)\Delta t,\\ Y(x_j,t)&=Y(x_j,t-\Delta t)+v(x_j,t-\Delta t)\Delta t. \end{aligned}$$

The values of $X(\xi_i,t)$, $Y(\xi_i,t)$ are obtained by interpolation using neighbouring values of $X(x_j,t)$, $Y(x_j,t)$. In this paper we have used a four-point Lagrangian interpolation.

In writing the quadrature rules for (u, v) use has been made of the fact that the strength of a vortex filament remains invariant with time. We also note that for a given N, $[u(x, t), v(x, t)]$ will be exact so long as, $G(x)H_1(x, \xi, t)/w(x)$ and $G(x)H_2(x, \xi, t)/w(x)$ are each polynomials of degree $2N$ or less in x. If at any time $t=T$ this condition is violated, then the results may be in error for $t>T$. Therefore it is advisable to begin with a sufficiently large value of N at $t=t_0$.

5. NUMERICAL EXAMPLES

The effectiveness of the numerical scheme devised above may be judged by the fact that it reproduces the test cases (a) and (b) of Section 3 exactly. The numerical results for the test case (c)

Table I. Comparison of numerically calculated vortex sheet shape at $t=0{\cdot}0$, 0·2, 0·6 for $(1-x^2)^{1/2}G(x)=3x(1-x^2)$ along $-1\leqslant x\leqslant 1$, $y=0$ at $t=0$ with van Dyke's[2] analytical time series solution (right half of vortex sheet only)

	Calculated from equation (17)		van Dyke	
t	X	Y	X	Y
0·00	0·00	0·00	0·00	0·00
	0·1564	0·00	0·1564	0·00
	0·3090	0·00	0·3090	0·00
	0·4540	0·00	0·4540	0·00
	0·5878	0·00	0·5878	0·00
	0·7071	0·00	0·7071	0·00
	0·8090	0·00	0·8090	0·00
	0·8910	0·00	0·8910	0·00
	0·9510	0·00	0·9510	0·00
	0·9877	0·00	0·9877	0·00
0·20	0·00	−0·1478	0·00	−0·1478
	0·1629	−0·1405	0·1629	−0·1404
	0·3199	−0·1193	0·3199	−0·1192
	0·4658	−0·0863	0·4658	−0·0863
	0·5959	−0·0452	0·5960	−0·0452
	0·7073	−0·0001	0·7073	0·0000
	0·7981	0·0448	0·7981	0·0447
	0·8677	0·0847	0·8677	0·0847
	0·9164	0·1161	0·9164	0·1162
	0·9451	0·1361	0·9452	0·1363
0·60	0·00	−0·4053	0·00	−0·3892
	0·2037	−0·3829	0·1974	−0·3674
	0·3901	−0·3185	0·3830	−0·3055
	0·5447	−0·2211	0·5427	−0·2141
	0·6619	−0·1023	0·6613	−0·1073
	0·7270	0·0611	0·7264	0·0000
	0·7469	0·1536	0·7351	0·0957
	0·6945	0·1670	0·6982	0·1721
	0·6853	0·1633	0·6394	0·2260
	0·6794	0·1624	0·5879	0·2575

Note. At $t=0{\cdot}60$ the van Dyke solution is not expected to be accurate owing to large truncation errors. The calculated solutions are presumably more accurate as they were obtained with $N=40$, $\Delta t=0{\cdot}0001$.

are shown in Table I for $t=0{\cdot}0$, 0·2, 0·6. At $t=0{\cdot}0$, 0·2 our numerical calculations coincide with van Dyke's analytical time series results. The numerical solution was then carried forward to $t=0{\cdot}6$, and, as expected, there are differences between the two calculations. It should be noted that van Dyke's results are good only for small values of t and are expected to become inaccurate due to truncation errors for large t such as $t=0{\cdot}6$. Our results, in comparison, are expected to be more accurate as they were obtained with $N=40$ and $\Delta t=0{\cdot}0001$, especially when one views the smoothness of $C(t)$ at $t=0{\cdot}6$ for which $N=40$ appears to be more than sufficient. (We remark here that $N=40$ means that $G(x)H_1(x, \xi, t)/w(x)$ and $G(x)H_2(x, \xi, t)/w(x)$ are each assumed to be representable by a polynomial in x of degree 80 or less.)

The numerical results for the test case (d) are shown in Table II for $t=0{\cdot}0$, 0·10. Once again the numerical results coincide with the analytical time series solution, further testifying to the accuracy of the method.

It is interesting to note that the point vortex method and that of Fink and Soh all fail right at $t=0$ itself!

Some computer runs were made replacing the Euler time-marching scheme with a fourth-order Runge–Kutta scheme. However, the Runge–Kutta scheme did not provide discernibly better

Table II. Comparison of numerically calculated vortex sheet shape at $t=0{\cdot}0$, 0·1 for $(1-x^2)^{1/2}G(x)=x$ along $-1\leqslant x\leqslant 1$, $y=\varepsilon(1-x^2)$ at $t=0$ with an analytical time series solution (right half of sheet only); $\varepsilon=-0{\cdot}10$

	Calculated from equation (17)		Analytical (case (d), Section 3)	
t	X	Y	X	Y
0·00	0·00	−0·1000	0·00	−0·1000
	0·1564	−0·0976	0·1564	−0·0976
	0·3090	−0·0905	0·3090	−0·0905
	0·4540	−0·0794	0·4540	−0·0794
	0·5878	−0·0655	0·5878	−0·0655
	0·7071	−0·0500	0·7071	−0·0500
	0·8090	−0·0345	0·8090	−0·0345
	0·8910	−0·0206	0·8910	−0·0206
	0·9511	−0·0095	0·9511	−0·0095
	0·9877	−0·0024	0·9877	−0·0024
0·10	0·00	−0·1495	0·00	−0·1498
	0·1580	−0·1470	0·1580	−0·1473
	0·3121	−0·1398	0·3121	−0·1400
	0·4584	−0·1285	0·4584	−0·1287
	0·5935	−0·1143	0·5935	−0·1145
	0·7140	−0·0986	0·7140	−0·0988
	0·8168	−0·0828	0·8168	−0·0830
	0·8996	−0·0686	0·8996	−0·0688
	0·9601	−0·0574	0·9601	−0·0576
	0·9971	−0·0501	0·9971	−0·0504

Note. The calculated results were obtained with $N=20$ and $\Delta t=0{\cdot}001$. Results with $\varepsilon=0{\cdot}10$ show similarly excellent comparisons at $t=0{\cdot}0$ and 0·1. The comparison further improves with smaller values of Δt.

results for a given CPU time on a computer when compared with the Euler scheme. The Euler scheme was therefore retained for its simplicity.

6. CONCLUSIONS

A numerical scheme for calculating the motion of a two-dimensional vortex sheet has been developed for a class of vorticity distribution on the sheet which satisfies Lighthill's lemma. The scheme accounts for the curvature of the vortex sheet as it develops and also correctly accounts for the singularities of the problem.

APPENDIX: A NOTE ON VORTEX SHEET SINGULARITIES

Lighthill's lemma does not shed any light as to the nature of the self-induced convection velocity of a vortex sheet for other types of vorticity distributions. In particular, it is not yet clear if there are other $G(x)$ which, while not satisfying the regularity condition on $G(x)$ as required by the lemma, would still produce a regular self-induced convection velocity on the sheet. The answer is probably no. A large number of functions $G(x)$, violating the regularity condition of the lemma, were selected and each showed a singularity at the sheet edges. We cite here two examples for the purposes of demonstration.

(a) $G(x)=1$ *along* $-1<x<1,\quad y=0$ *at* $t=0$

For this distribution, at $t=0$, we have

$$u(x,0)=0, \qquad v(x,0)=(1/2\pi)\ln[(1+x)/(1-x)].$$

Thus v has a logarithmic singularity at the sheet edges $x=\pm 1$, $y=0$. It is rather easy to extend and show that, in general, if $G(x)$ has a polynomial representation in x, v will still be logarithmically singular at the sheet edges.

(b) $G(x)=\ln[(1+x)/(1-x)]/(1-x^2)^{1/2}$ *along* $-1<x<1,\quad y=0$ *at* $t=0$

For this distribution, at $t=0$, we have

$$u(x,0)=0, \qquad v(x,0)=-(\pi/2)(1-x^2)^{-1/2}.$$

Thus v has a square-root singularity at the sheet edges $x=\pm 1$, $y=0$. Here too we can show that if

$$G(x)=x^n\ln[(1+x)/(1-x)]/(1-x)^2)^{1/2},$$

where n is a positive integer, then the square-root singularity will persist at the sheet edges.

In the examples (a) and (b), for the given $G(x)$, even if the sheet at $t=0$ were a smooth curve but not flat as chosen, their respective singularities at the sheet edges would have remained. However, if the sheet itself has kinks, then additional singularities in (u, v) will generally occur at each of the kinks.

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